University of Wisconsin – Eau Claire
2025 Math meet
February 15, 2025

Event 1: Elementary Algebra

Name: ____________________ **School:** ____________________ **Team:** ____________________

Simplify final answers and place them in the space given.

(1) (2 points) A restaurant has 25 square tables that can seat 1 person on each side of the square. For a large party, they decide to put the tables side-by-side to create one long table. When two square tables are set side-by-side, no one can sit at the side where the tables are connected. How many people can sit at the resulting long table?

Solution: The long table consists of 23 square tables that are on the interior (sharing an edge with each of 2 other tables) and 2 square tables that are on the ends (sharing an edge with just 1 other table). Each interior table can seat 2 people, and each end table can seat 3 people. Therefore, the total number of people who can sit is $2 * 23 + 3 * 2 = 52$.

Answer: 52

(2) (3 points) The sum of three distinct numbers is 27. The largest minus the smallest is 6. The second largest minus the smallest is 3. What are the three numbers?

Solution: Call the three numbers x, y, z with $x > y > z$. Then $x - z = 6$, so $x = z + 6$. And $y - z = 3$, so $y = z + 3$. The sum is $x + y + z = 27$ and this can be rewritten as $(z + 6) + (z + 3) + z = 27$. By collecting like terms, we get $3z + 9 = 27$, so $3z = 18$, so $z = 6$ is the smallest number. This means that the three numbers are 6, 9, and 12.

Answer: 6, 9, 12

(3) (4 points) Alice has 16 marbles while Bob has an unknown number of marbles. Alice gives Bob half of her marbles. Then Bob counts up how many marbles he has and gives half of them to Alice. If Alice ends this transaction with 16 marbles then how many marbles did Bob have at the start?

Solution: Call Bob's starting number of marbles B. After Alice gives Bob half her marbles, Alice has 8 marbles and Bob has $B + 8$. After Bob gives Alice half his marbles, Alice has $8 + \frac{B+8}{2}$. Then we know $8 + \frac{B+8}{2} = 16$, so $\frac{B+8}{2} = 8$, so $B + 8 = 16$. Therefore $\boxed{B = 8}$.

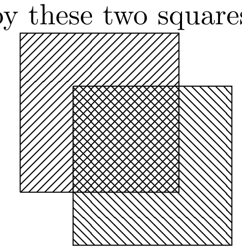
Answer: 8

Event 2: Geometry

Name: ____________________ **School:** ____________________ **Team:** ____________________

Simplify final answers and place them in the space given.

(1) (2 points) Two 3×3 squares overlap in a 2×2 square. What is the total area covered by these two squares?



Solution: The double counting principle (also known as the inclusion-exclusion principle) says that the area will be

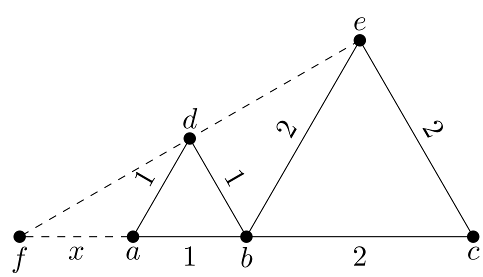
$$(\text{Area of } 1^{\text{st}} \text{ square}) + (\text{Area of } 2^{\text{nd}} \text{ square}) - (\text{Area of overlap}) \\ = 9 + 9 - 4 = 14$$

Answer: 14

(2) (3 points) Consider the diagram below. Δ_{abd} is an equilateral triangle with side length 1. Δ_{cbe} is equilateral of side length 2. The points f, d, e are all colinear. The points f, a, b, c are all colinear. Find the length of the edge from f to a.

Solution: The triangles Δ_{fce} and Δ_{fbd} are similar triangles. In order to see that they are similar, notice that $\angle abd$ and $\angle bce$ must both be $60°$, as Δ_{abd} and Δ_{bce} are both equilateral. Δ_{fce} and Δ_{fbd} now have two angle measures in common, and so they are similar.

By looking at their rightmost edges, Δ_{fce} has sides double the length of Δ_{fbd}'s sides. As a consequence, $x + 1 + 2 = 2 \cdot (x + 1)$. Simplifying, $x + 3 = 2x + 2$. Finally solving, $1 = x$

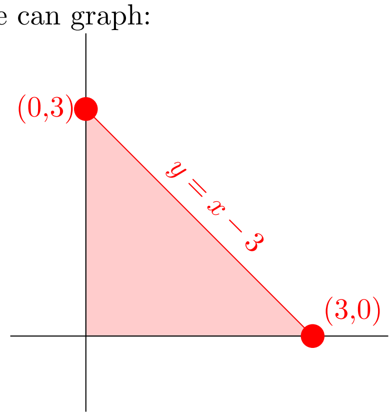


Answer: 1

(3) (4 points) Find the perimeter of the region in the xy-plane that consists of the points (x, y) such that $|x| + |y| \leq 3$. Express your answer in simplified radical form.

Solution: This region is symmetric over the x-axis and over the y-axis, since the absolute values mean that it does not matter if x and y are positive or negative. This means we can compute the perimeter of the region in quadrant 1 and then multiply by 4.

When x and y are positive the inequality becomes $x + y \leq 3$, or $y \leq x - 3$. This we can graph:



The length of this line is $\sqrt{3^2 + 3^2} = 3\sqrt{2}$. Now multiply by 4 to get $\boxed{\text{perimeter} = 12\sqrt{2}}$.

Answer: $12\sqrt{2}$.

Event 3: Intermediate Algebra

Name: ____________________ **School:** ____________________ **Team:** ____________________

Simplify final answers and place them in the space given.

(1) (2 points) In the table below, each unshaded square is given by adding together the two unshaded squares immediately below. For example, the "11" appears since $5+6=11$. What must x be?

			60			
	11					
5		6		x		7

Solution: We proceed by filling in the table using the provided rules

			60			
		$x+17$		$2x+13$		
	11		$6+x$		$x+7$	
5		6		x		7

So we must have $60 = 3x + 30$, which is solved by $\boxed{x = 10}$.

Answer: 10

(2) (3 points) How many pairs of consecutive positive integers have a product less than 300? (Order does not matter in this problem, so $8 \cdot 7$ and $7 \cdot 8$ count as the same pair.) Keep in mind that we are looking for the number of such pairs, not a listing of all such pairs.

Solution: We want to count the number of choices of positive integers n with $n(n+1) < 300$. We could do so by solving $x(x+1) = 300$ and then deciding whether we want to round up or down, but without a calculator using the quadratic formula will be difficult. Instead we will check for the first natural number n with $n(n+1) \geq 300$. Start with multiples of 10, since they are easy to compute: $20 \cdot 21 = 420$ and $10 \cdot 11 = 110$, so this threshold is somewhere in between 10 and 20. We may now start at the midpoint ($n = 15$) and proceed with our search. $15 \cdot 16 = 240$; $16 \cdot 17 = 170 + 102 = 272$, $17 \cdot 18 = 306$. Thus, $n = 17$ is the first natural number with $n(n+1) > 300$.

Conclusion: The options for n are $1, 2, 3, \ldots, 16$. There are $\boxed{\text{16 possibilities}}$.

Answer: 16

(3) (4 points) If $5 \cdot \log_5(x) = \log_5(5x)$ then what is $\log_5(x)$? (Give a precise numerical value for $\log_5(x)$, an answer which depends on x is not acceptable.)

Solution: Via logarithm laws, the right hand side of $5 \cdot \log_5(x) = \log_5(5x)$ simplifies as $\log_5(5x) = \log_5(5) + \log_5(x) = 1 + \log_5(x)$. This leaves us with $5 \cdot \log_5(x) = \log_5(x) + 1$, which further simplifes to $4 \log_5(x) = 1$. Now divide both sides by 4 to get $\boxed{\log_5(x) = \frac{1}{4}}$

Answer: $\frac{1}{4}$ or 0.25

Event 4: Advanced Mathematics

Name: ____________________ **School:** ____________________ **Team:** ____________________

Simplify final answers and place them in the space given.

(1) (2 points) An arithmetic progression is a sequence of three integers $x < y < z$ which satisfy that $z - y = y - x$, so for example $(3, 5, 7)$ is an arithmetic progression. How many arithmetic progressions can be chosen from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$?

Solution: By solving for y we get that any arithmetic triple is $\left(x, \frac{x+z}{2}, z\right)$ with $1 \leq x < z \leq 10$ and and the added stipulation that $x + z$ has to be even (since y has to be a whole number). This means that when x is even, z is even and when x is odd z is odd. List the possibilities of z for each choice of x

$x = 1:$	$z = 3, 5, 7, 9$	4 possibilities
$x = 2:$	$z = 4, 6, 8, 10$	4 possibilities
$x = 3:$	$z = 5, 7, 9$	3 possibilities
$x = 4:$	$z = 6, 8, 10$	3 possibilities
$x = 5:$	$z = 7, 9$	2 possibilities
$x = 6:$	$z = 8, 10$	2 possibilities
$x = 7:$	$z = 9$	1 possibility
$x = 8:$	$z = 10$	1 possibility
$x = 9$ or $10:$		0 possibilities

In total, we have $4 + 4 + 3 + 3 + 2 + 2 + 1 + 1 = \boxed{20}$ arithmetic progressions.

Answer: 20

Challenge problem for further thought. Don't stop at 10. How many arithmetic progressions are there in $\{1, 2, 3 \ldots, n\}$ for any n?

(2) (3 points) A restaurant has an unlimited number of square tables (which can seat 1 person on each side) and rectangular tables (which can seat 1 person on each of the two short sides and 2 people on each of the two long sides). Each side of the square tables is the same length as the short sides of the rectangular tables. The restaurant wants to put the tables side-by-side to create one long table that seats 18 people. How many ways are there to do this? (Different orders of tables, such as square-rectangle and rectangle-square, count as different ways.)

Solution The long table will seat 2 people on the ends and 16 people on the long sides. The two long sides can seat an equal number of people, so we want to count the number of ways to arrange the tables so that we have the capacity to seat 8 people on each side, where square tables seat 1 person and rectangular tables seat 2 on that side. Let F_n be the number of ways to line up a bunch of square (S) and rectangular (R) tables to build a long table setaing n people on a long side. Then the first few values of F_n are

n	F_n	Ways to build this table: S=Square, R=Rectangle
1	1	S
2	2	SS, R
3	3	SSS, RS, SR
4	5	SSSS, RR, RSS, SRS, SSR

This looks like the Fibonacci sequence. In fact, it *is* the Fibonacci sequence: To seat n people on one side, you can take an arrangement of tables that seats $n-1$ and add a square table on the end, or you can take an arrangement that seats $n-2$ and add a rectangular table. Therefore, $F_n = F_{n-1} + F_{n-2}$. This means that the table above can be extended as follows:

n	F_n
5	8
6	13
7	21
8	34

Answer: 34

(3) (4 points) What is the average value amongst all of the solutions to

$$|x+1|^5 - 5|x+1|^3 + 5|x+1| - 1 = 0?$$

Solution: Finding a solution to this equation is equivalent to finding an x-intercept to the graph of $y = |x+1|^5 - 5|x+1|^3 + 5|x+1| - 1$, so we will think graphically, even though we do not have the tools to directly graph it.

Solution: If you delete the "+1"s in the absolute value bars, then you get $y = |x|^5 - 5|x|^3 + 5|x| - 1$, which is an even function. This means that if $(C, 0)$ is one is one x-intercept to $y = |x|^5 - 5|x|^3 + 5|x| - 1$, then so is $x = (-C, 0)$. The "+1" shifts the graph left by 1. This means that $(C-1, 0)$ and $(-C-1, 0)$ are x-intercepts. These x-coordinates average to -1. As we have grouped our solutions into pairs that average to -1, the average of all solutions will be $\boxed{-1}$.

(A subtle point) Maybe you are concerned that there might be no solutions. In this case the average will not make sense and "does not exist" would be the best answer. To avoid this notice that when $x = -1$, you will get $y = -1 < 0$. When $x = 9$, you will get $y \approx 10^5 > 0$. The graph will need to pass through the x-axis, and so there will be at least one solution.

Answer: -1

Team Event

School: ____________________ **Team:** ____________________
Simplify final answers and place them in the space given.

(1) (10 points) Let N be the largest four-digit number such that all four digits are distinct, and N is divisible by each of its digits. What is N?

Solution: If we can find a solution with 9 in the thousands, 8 in the hundredths and 7 in the tens then that solution will be the largest. Doing long division: $\frac{9870}{9} = 1096 + \frac{6}{9}$. The remainder of 6 means that we need to ones digit to be 3 in order to get a multiple of 9. So 9873 is the option we are stuck with. But, 9873 is not even, so it cannot be a multiple of 8. Since we cannot find a solution of the form $987\star$ we look for the next largest option, $986\star$.

Again doing long division, $\frac{9860}{9} = 1095 + \frac{5}{9}$, the remainder of 5 forces us to use 9864. We now check for divisbility by 8: $\frac{9864}{8} = 1233$. Disibility by 8 and 9 implies divisibility by 6 and 4, so there is no need to check them.

Answer: 9864

Challenging problem for further thought. *How many* 4 digits numbers exist which are divisible by each of their digits?

(2) (10 points) You start putting numbers into a table following the pattern below:

	1^{st} column
1^{st} row	1

	1^{st} column	2^{nd} column
1^{st} row	1	3
2^{nd} row	2	

	1^{st} column	2^{nd} column	3^{rd} column
1^{st} row	1	3	6
2^{nd} row	2	5	
3^{rd} row	4		

	1^{st} column	2^{nd} column	3^{rd} column	4^{th} column
1^{st} row	1	3	6	10
2^{nd} row	2	5	9	
3^{rd} row	4	8		
4^{th} row	7			

In which row does 2025 appear?

Solution: Notice that the number in 1^{st} row, n^{th} column is the n^{th} triangular number, which is given by $\frac{n(n+1)}{2}$. We start by solving $\frac{x(x+1)}{2} = 2025$. Multiplying both sides by 2 and distributing yeilds $x^2 + x = 4050$. We now solve this by the quadratic formula $x = \dfrac{-1 \pm \sqrt{16201}}{2}$. We need only consider the positive solution, which is about $x \approx 63.1$.

As a consequence the entry in the 1^{st} row, 63^{th} column is $\dfrac{63 \cdot 64}{2} = 2016$. We are now only 9 away from 2025. Start at the next entry and count up. (We indicate the row an column in subscripts:)

			$2025_{56,9}$
		$\iddots$	
	$2018_{63,2}$		
$2017_{64,1}$			

Thus 2025 appears in the 56^{th} row.

Answer: 56

(3) (10 points) Determine the exact value of

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}}$$

Solution. We start by giving this number a name. Call it x. Notice that we see the formula for x inside of the formula for x. Thus,

$$x = 2 + \frac{1}{x}$$

We solve for x. Multiply both sides by x and move everything onto the same side:

$$x^2 - 2x - 1 = 0$$

Now use the quadratic formula and simplify

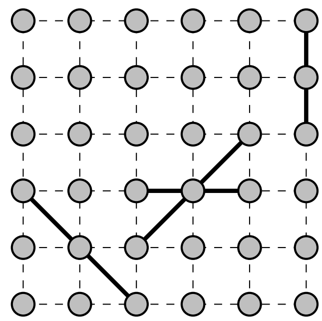
$$x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$$

As there are no minus signs in the formula for x, we record the positive solution, $\boxed{1 + \sqrt{2}}$

Answer: $1 + \sqrt{2}$

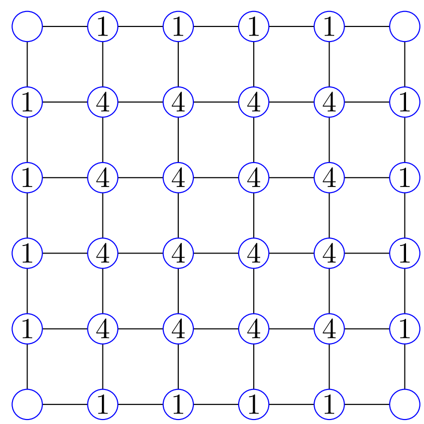
Suggested continued reading. This is an example of a *continued fraction.* These come up and have important roles in analysis and number theory.

(4) (10 points) On a 6×6 grid you start drawing horizontal, vertical and diagonal lines each through three consequitive grid points either vertically, horizontally or diagonally (with slope$=\pm 1$). The lines are allowed to cross each other. For example, four such lines appear in the diagram below. How many such lines can be drawn on this grid?



Solution. In order to specify a line in this problem, we may record its midpoint, and whether the line is vertical, horizontal, diagonal with positive slope or diagonal with negative slope.

The corner points in the grid cannot be midpoints of any line, otherwise the line will escape the grid. The non-corner points on the vertical edges of the square are only midpoints of vertical lines. The non-corner points on the horizontal edges of the square are only midpoints of horizontal lines. All of the interior points can be midpoints of any of the four types of line. Thus, if we ask each vertex how many lines it is the midpoint of then we get the diagram to the right:



This totals up to $1 \cdot 16 + 4 \cdot 16 = 5 \cdot 16 = \boxed{80}$ lines

Answer: 80

Challenging problem for further thought What if we use lines of length 2, 4, or 5?

(5) (10 points) Four of the vertices of a regular octagon are colored red, and the remaining four are colored green. We agree that two such colorings are the same if we can obtain one from the other by a rotation of the octagon. Note that a reflection (or flip) of the octagon is not a rotation. How many distinct colorings of the octagon are there?

Solution: (Really understand this solution will require you to draw some pictures. Get some scratch paper out and then read along.) Number the vertices of the octagon from 1 to 8 in clockwise order. The symbol $[a, b, c, d]$ will denote a coloring whose red vertices are numbered by a, b, c, and d, with $a < b < c < d$. We sort our work according to how the red vertices are grouped together. Any coloring with no adjacent red vertices is the same as $[1, 3, 5, 7]$ (up to rotation). This gives one coloring so far.

Next we think about the case that two red vertices are adjacent while the other two are not. Up to rotation we may as well assume the two adjacent red vertices are indexed 1 and 2. In this case vertices 8 and 3 must be blue. We can now just list the possibilities, while remembering that the other two red vertices cannot be adjacent in this case.

$$[1, 2, 4, 6], [1, 2, 4, 7], [1, 2, 5, 7]$$

This gives three more colorings

Next we think about the case that there are two groups of two adjacent red vertices. We again assume that 1 and 2 are red, so that 8 and 3 are is blue. We now list cases:

$$[1, 2, 4, 5], [1, 2, 5, 6], [1, 2, 6, 7]$$

$[1, 2, 4, 5]$ and $[1, 2, 6, 7]$ are the same up to rotation, so This gives two more colorings

Next we list the coloring consisting of three adjacent red vertices (up to rotation, 1,2,3) and one other red vertex (either 5, 6,or 7).

$$[1, 2, 3, 5], [1, 2, 3, 6], [1, 2, 3, 7]$$

This gives three more colorings

Finally, if all four red vertices are adjacent, then we may rotate to get $[1, 2, 3, 4]$. This gives one more coloring.

The total number of colorings is $\boxed{1 + 3 + 2 + 3 + 1 = 10}$.

Answer: 10

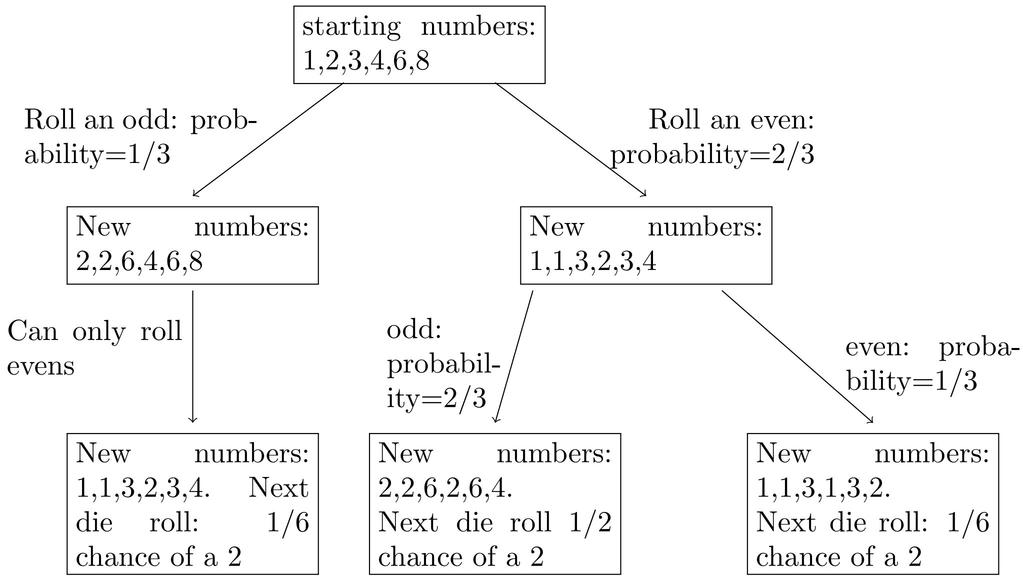
Challenging problem for further thought. Go past 8. What about a 10-gon (a polygon with 10 sides)? A 12-gon? A $2n$-gon for any n?

(6) (10 points) A die starts with the numbers 1, 2, 3, 4, 6, and 8 on its faces. Do the following:

- Roll the die. If an odd number appears on the top of the die, then multiply all odd numbers on the die by 2. If an even number appears on the top of the die then divide all even numbers on the die by 2.
- Roll the die again. If an odd number appears on the top of the die, then multiply all odd numbers on the die by 2. If an even number appears on the top of the die then divide all even numbers on the die by 2.
- Roll the die one last time and record what number appears on the top of the die.

What is the probability that the third number rolled is a 2? Give your answer as a fraction in reduced form.

Solution: On the first roll of the die there is a $2/6 = 1/3$ chance of rolling an odd and a $4/6 = 2/3$ change of rolling an even, per the tree below



Coming from the left most branch we get a $\frac{1}{3} \cdot \frac{1}{6} = \frac{1}{18}$ chance of a 2.
From the center branch: we get a $\frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{2} = \frac{2}{9}$ chance of a 2.
From the rightmost branch: we get a $\frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{6} = \frac{1}{27}$ chance of a 2.
Add these up to get a $\frac{1}{18} + \frac{2}{9} + \frac{1}{27} = \boxed{\frac{17}{54}}$ chance of a 2 on the third roll.

Answer: $\underline{17/54}$

Challenging problem for further thought You keep rolling the die and iterating these instructions. To what value does the probability of rolling a 2 converge?